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On an invariant property of water waves

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The discussion concerns free wave motions generated from rest in a finite region of an ocean of heavy liquid lying on a horizontal plane. It is shown that the horizontal first moment of the free-surface displacement varies linearly with time. Hence, if the total volume displaced is not zero and therefore the centroid of the displacement is definable, the centroid travels with a constant horizontal velocity as the wave motion evolves. This conclusion holds exactly for waves of any amplitude and even remains applicable subsequent to the breaking of waves.

The purpose of this note is to demonstrate an exact property of water waves which appears not to have been noticed previously, although a corresponding result was derived on the basis of long-wave approximations by Keulegan & Patterson (1940). The property is especially remarkable in that it does not depend on the wave motion being small or irrotational.

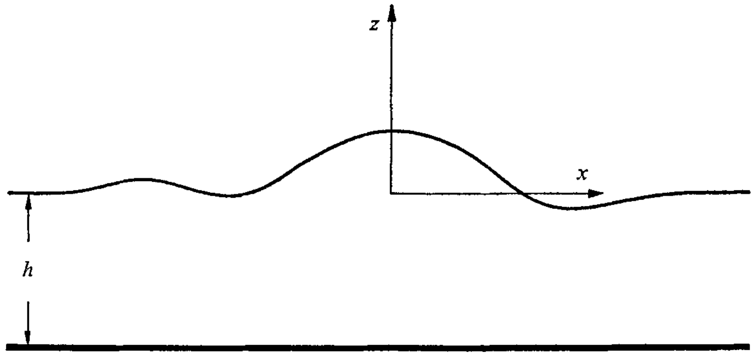

FIGURE 1. Definition sketch.

We consider waves formed in an ocean of incompressible liquid whose depth when undisturbed is a constant h, which may be infinite (see figure 1). Axes (x, y, z) are taken with their origin in the undisturbed free surface and with z vertical upwards, so that the bottom is represented by $z = -h$. The equation of the free surface is written

$$z = \zeta(x, y, t), \tag{1}$$

where t is time. It is assumed that the wave motion is generated in some finite

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region over a finite time, during which some liquid may be added or removed, and subsequently the system is free from horizontal external forces. Thus no frictional force along the bottom is allowed, but the following argument applies precisely to a viscous as well as to an inviscid liquid in the case of infinite depth. We take for granted that, at finite times, the flow field decays at least more rapidly than a dipole field at sufficiently large distances from the centre of the disturbance.

The moment of the free-surface displacement is defined by

$$\mathbf{M} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{r}\zeta\, dx\, dy, \tag{2}$$

in which $\mathbf{r} = (x, y)$ is the horizontal position vector. Accordingly, if the total volume displaced, i.e.

$$Q = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \zeta\, dx\, dy, \tag{3}$$

is not zero, the position of the centroid of the displacement is given by

$$\mathbf{R} = \mathbf{M}/Q. \tag{4}$$

Our object is to show that $d\mathbf{M}/dt$ is a constant vector, and hence so is the velocity $d\mathbf{R}/dt$ of the centroid.

The case of two-dimensional motions, say in the plane (x, z), is treated first for simplicity of illustration, and then the argument is extended to the three-dimensional problem.

Two-dimensional motions

For these there exists a stream function $\psi(x, z, t)$ such that the velocity components parallel to x and z are

$$(u, w) = \left(-\frac{\partial\psi}{\partial z}, \ \frac{\partial\psi}{\partial x}\right). \tag{5}$$

On the bottom $z = -h$ where $w = 0$, the value of ψ is a constant which we may take to be zero. The kinematical condition at the free surface $z = \zeta(x, t)$ is

$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} - w = 0, \tag{6}$$

which in the light of (5) is seen to be equivalent to

$$\frac{\partial\zeta}{\partial t} - \frac{\partial\psi_s}{\partial x} = 0, \tag{7}$$

where
$$\psi_s(x, t) = \psi\{x, \zeta(x, t), t\} \tag{8}$$

is just the value of ψ at the free surface. The two-dimensional form of the definition (2) is

$$M = \int_{-\infty}^{\infty} x\zeta\, dx,$$

from which we obtain, using (7),

$$\frac{dM}{dt} = \int_{-\infty}^{\infty} x\frac{\partial\zeta}{\partial t}\, dx = \int_{-\infty}^{\infty} x\frac{\partial\psi_s}{\partial x}\, dx.$$

After an integration by parts the integrated terms vanish because ψ_s is by assumption sufficiently small as $|x| \to \infty$, and there follows

$$\frac{dM}{dt} = -\int_{-\infty}^{\infty} \psi_s \, dx. \tag{9}$$

Now, the total horizontal momentum, or impulse, of the liquid per unit span is given by

$$I = \int_{-\infty}^{\infty} \int_{-h}^{\zeta} \rho u \, dz \, dx = -\int_{-\infty}^{\infty} \int_{-h}^{\zeta} \rho \frac{\partial \psi}{\partial z} \, dz \, dx = -\int_{-\infty}^{\infty} \rho \psi_s \, dx, \tag{10}$$

where ρ is its (constant) density. Thus we have $dM/dt = I/\rho$. But I is a constant in the absence, as supposed, of horizontal external forces. This fact may, of course, be confirmed readily from the equations of motion coupled with the assumption that the only external stress on the free surface is a constant pressure (cf. Batchelor 1967, §3.2). The overall momentum balance is unaffected by surface tension, which evidently produces no net horizontal force on a surface that is asymptotic at both ends to the same horizontal plane.† Allowing also for viscosity of the liquid, we may still assert that I is constant if $h = \infty$.

The total volume displaced by the free surface, per unit span, is

$$Q = \int_{-\infty}^{\infty} \zeta \, dx,$$

and is obviously constant, as shown by (7). If $Q \neq 0$, the x co-ordinate of the centroid is defined by $X = M/Q$, and we conclude from the above that

$$\frac{dX}{dt} = \frac{1}{Q}\frac{dM}{dt} = \frac{I}{\rho Q} = \text{const.}, \tag{11}$$

which is the anticipated result for the two-dimensional case. Note that the velocity given by (11) is unrelated to the gravitational constant g, although the evolution of the wave form will depend essentially on g.

Three-dimensional motions

The theory for this case may be made closely analogous to the preceding. By virtue of the incompressibility of the liquid, the velocity field is solenoidal and so is expressible as the curl of a vector potential, which may be tailored to the conditions of the problem by including the gradient of an arbitrary scalar. Specifically, in order that it should play the same rôle as the stream function in the two-dimensional theory, the vector potential is required to vanish on the bottom and to be suitably small at infinity. It is a simple matter to verify that

$$\mathbf{A} = \left(\int_{-h}^{z} v(x,y,z',t) \, dz', -\int_{-h}^{z} u(x,y,z',t) \, dz', 0 \right) \tag{12}$$

has these properties (even, in fact, if the bottom is not everywhere horizontal), and that

$$\mathbf{u} = (u, v, w) = \nabla \times \mathbf{A} \tag{13}$$

in consequence of the condition $\nabla . \mathbf{u} = 0$.

† The fact that surface tension is immaterial to our conclusions was kindly pointed out to us by a referee.

Expressed in terms of the components $A^{1,2}$ of this vector potential, the kinematical condition at the free surface, described by (1), is

$$\frac{\partial \zeta}{\partial t} - \frac{\partial A^2}{\partial z}\frac{\partial \zeta}{\partial x} + \frac{\partial A^1}{\partial z}\frac{\partial \zeta}{\partial y} - \left(\frac{\partial A^2}{\partial x} - \frac{\partial A^1}{\partial y}\right) = 0, \tag{14}$$

which is equivalent to

$$\frac{\partial \zeta}{\partial t} - \frac{\partial A^2_s}{\partial x} + \frac{\partial A^1_s}{\partial y} = 0, \tag{15}$$

where

$$A^{1,2}_s(x,y,t) = A^{1,2}\{x,y,\zeta(x,y,t),t\}. \tag{16}$$

Using (15), we now obtain from (2)

$$\frac{d\mathbf{M}}{dt} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x,y)\frac{\partial \zeta}{\partial t}\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x,y)\left\{\frac{\partial A^2_s}{\partial x} - \frac{\partial A^1_s}{\partial y}\right\}dx\,dy,$$

which after integrations by parts reduces to

$$\frac{d\mathbf{M}}{dt} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(-A^2_s, A^1_s)\,dx\,dy. \tag{17}$$

The horizontal impulse of the liquid is given by

$$\mathbf{I} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-h}^{\zeta}\rho(u,v)\,dz\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-h}^{\zeta}\rho\left(-\frac{\partial A^2}{\partial z}, \ \frac{\partial A^1}{\partial z}\right)dz\,dx\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(-A^2_s, A^1_s)\,dx\,dy, \tag{18}$$

since $\mathbf{A}$ vanishes on the bottom. Thus we have

$$\frac{d\mathbf{M}}{dt} = \frac{\mathbf{I}}{\rho}, \tag{19}$$

and this is a constant vector in the absence of horizontal external forces. Again, as in the two-dimensional problem, surface tension evidently produces no net force capable of affecting $\mathbf{I}$. If the constant Q given by (3) is not zero, so that the centroid of the free-surface displacement is defined by (4), it follows that

$$\frac{d\mathbf{R}}{dt} = \frac{1}{Q}\frac{d\mathbf{M}}{dt} = \frac{\mathbf{I}}{\rho Q} = \text{const.} \tag{20}$$

This means that as the free wave motion evolves the centroid moves with a constant horizontal velocity, which is originally fixed in both direction and magnitude by the forces initiating the wave motion.

It is worth further emphasis that the property thus demonstrated is independent of several approximations often used in water-wave theories. The motion of the liquid need not be irrotational, nor need the wave amplitude be small. Surface tension may be present, and the liquid may be viscous provided it is deep enough for shear forces along the bottom to be insignificant. Also, the free-surface displacement ζ need not everywhere be a single-valued function of the position vector $\mathbf{r}$. In the two-dimensional case, if ζ and ψ_s are multiple-valued, the evaluation of integrals such as (10) can proceed unambiguously following the transformation $dx = (\partial x/\partial l)\,dl$, where l is arc length along the free surface; and an obvious extension of this device is applicable to the three-dimensional case, confirming the results as given above, It appears, therefore, that these results will remain true subsequent to the breaking of waves.

Note that the derivation of the equality (19) is just a matter of kinematics: the invariant property in question depends on dynamical principles only in that $\mathbf{I}$ is a constant vector. Accordingly it is easy to extend present ideas to cases where horizontal external forces influence the wave motion, as when the bottom is not everywhere horizontal. Letting $\mathbf{F}(t)$ denote the net horizontal force acting on the liquid, one may anticipate the formula

$$\frac{d^2\mathbf{M}}{dt^2} = \frac{1}{\rho}\frac{d\mathbf{I}}{dt} = \frac{\mathbf{F}}{\rho}, \tag{21}$$

which can readily be confirmed by detailed calculations.

Finally it is desirable to resolve what might appear to be a conflict between the present results and results derived by Wehausen & Laitone (1960, p. 508) on the basis of linearized surface-wave theory. Considering the two-dimensional problem they expressed $\zeta(x, t)$ as a superposition of two wave-packets, one comprised of waves travelling to the right and the other from waves travelling to the left, and they showed that the centroid of each moves with the group velocity of infinitely long waves. The centroid of the total displacement, however, has a velocity that depends on the division of the initial disturbance into right and left-travelling components. It is well known that an arbitrary (infinitesimal) displacement of the surface released from rest splits into two equal wave packets travelling in opposite directions, and so evidently the initial values of ζ do not affect dM/dt. In fact dM/dt is determined by the initial velocities $\partial\zeta/\partial t$ of the surface, which fix the initial velocity field in the fluid; and thus the motion of the centroid is related to the horizontal momentum imparted to the fluid, precisely as shown in the foregoing discussion. While being formally correct, the conclusions reached by Wehausen & Laitone give an extraneous view of the present issue.

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